A NOTE ON INTERPOLATION SERIES IN THE COMPLEX DOMAIN

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ABSTRACT. Given a convergent sequence of nodes we present a one-dimensional-holomorphic-function version of the Newton interpolation method of polynomials. It also generalises the Taylor and the Laurent formula. In other words, we present an effective identity theorem for complex domains.

1. INTRODUCTION

We all know well that there is exactly one polynomial of degree at most n that takes given values at a given set of $n + 1$ distinct points $c_1, \ldots, c_{n+1}$. Moreover, this polynomial is given by the following Newton interpolation formula, (see [1], Sec. 2.6)

$$P(x) = P(c_1) + \Delta^1 P(c_1, c_2)(z - c_1) + \ldots + \Delta^n P(c_1, \ldots, c_{n+1})(z - c_1) \cdots (z - c_n).$$

The so-called *devided differences* Δ^n are given by

$$\Delta^1 P(c_1, c_2) = \frac{P(c_2) - P(c_1)}{c_2 - c_1}, \quad \Delta^1 P(c_1, c_2, c_3) = \frac{P(c_1, c_3) - P(c_1, c_2)}{c_3 - c_2},$$

and so on, or equivalently

$$\Delta^{n-1} P(c_1, \ldots, c_n) = \sum_{1 \le k \le n} \frac{P(c_k)}{(c_k - c_1) \ldots \widehat{(c_k - c_k)} \ldots (c_k - c_n)}.$$

An analogical infinite series for a given infinit set of points c_n is known as the *interpolation series*, (see [2]).

We generalise the polynomial interpolation formula to give an interpolation series for a holomorphic function, thus establishing an effective identity theorem. In our approach the points c_i don't need to be distinct. In this way we also obtain a generalisation of Taylor or Laurant formula for series expansion of a holomorphic function.

Two more remarks are due. This result is probably known, but the author is not aware of it. The proofs are simple but somehow neat.

2. THE RESULT

Theorem 1. *Consider two open discs $D_1, D_2 \subset \mathbb{C}$ such that $\overline{D_1} \subset D_2$ with centres c and d, the curves Γ_1 and Γ_2 which go once around the respective discs in the positive direction and a holomorphic function $f : G \longrightarrow \mathbb{C}$, where $G \supset \overline{D_2} \setminus D_1$. Consider also sequences of complex numbers (c_n) and (d_n) convergent respectively*

to c and d and omitting $|\Gamma_2|$. Then the function f has the following representation as the sum of locally uniformly absolutely convergent series:

$$f(z) = \sum_{n\geq 1} a_{-n} \frac{1}{(z-d_1)\ldots(z-d_n)} + \sum_{n\geq 0} a_n (z-c_1)\ldots(z-c_n), \quad (z \in D_2 \setminus \overline{D_1})$$

where

$$a_{-n} = \frac{1}{2\pi i} \int_{\Gamma_1} f(\xi)(\xi - d_1)\ldots(\xi - d_{n-1}) \, d\xi \qquad (n \geq 1)$$

and

$$a_n = \frac{1}{2\pi i} \int_{\Gamma_2} \frac{f(\xi)}{(\xi - c_1)\ldots(\xi - c_{n+1})} \, d\xi. \qquad (n \geq 0)$$

Before we embark on the proof let us first consider the following

Lemma 1. *Given a sequence (λ_n) of complex numbers and a nonzero complex x the equality*

$$\frac{1}{x} = \sum_{n\geq 0} \frac{(\lambda_1 - x)\ldots(\lambda_n - x)}{\lambda_1 \ldots \lambda_n \lambda_{n+1}}$$

is satisfied if and only if

$$\prod_{n\geq 1} \frac{\lambda_n - x}{\lambda_n} = 0.$$

This is true for instance for any sequence (λ_n) of complex numbers such that $|\frac{\lambda_n - x}{\lambda_n}| < \theta < 1$ for sufficiently large n-s, and then the above series is absolutely convergent.

Proof. We even have

$$\frac{1}{x} = \sum_{n\geq 0} \frac{(\lambda_1 - x)\ldots(\lambda_n - x)}{\lambda_1 \ldots \lambda_n \lambda_{n+1}} + \frac{1}{x} \prod_{n\geq 1} \frac{\lambda_n - x}{\lambda_n},$$

provided that the sum or the product converges.

Indeed, this follows from the following identity

$$\sum_{0\leq n\leq k} \frac{(\lambda_1 - x)\ldots(\lambda_n - x)}{\lambda_1 \ldots \lambda_n \lambda_{n+1}} = \frac{1}{x} - \left(\prod_{1\leq n\leq k+1} \frac{\lambda_n - x}{\lambda_n} \right) \frac{1}{x}.$$

To arrive at this equality, observe that the left-hand side is equal to

$$\frac{1}{\lambda_1} + \frac{\lambda_1 - x}{\lambda_1} \left(\frac{1}{\lambda_2} + \frac{\lambda_2 - x}{\lambda_2} \left(\ldots \left(\frac{1}{\lambda_k} + \frac{\lambda_k - x}{\lambda_k} \left(\frac{1}{\lambda_{k+1}} \right) \right) \ldots \right) \right).$$

From this we obtain the right-hand side if we bear in mind that

$$\frac{1}{\lambda_{k+1}} = \frac{1}{x} - \frac{\lambda_{k+1} - x}{\lambda_{k+1}} \frac{1}{x}$$

and that for $\lambda \neq 0$ we have

$$\left[y \longmapsto \frac{1}{\lambda} + \frac{\lambda - x}{\lambda} y \right] = \left[\frac{1}{x} + h \longmapsto \frac{1}{x} + \frac{\lambda - x}{\lambda} h \right].$$

The second part of the lemma is obvious as (λ_n) is bounded away from 0 by at least $|x/2|$. $\qquad\square$

We are ready now to tackle Theorem 1.

Proof. In essence, we proceed as in the standard proof of Laurant series expandability of holomorphic functions in annular domains. We begin with the Cauchy theorem for the cycle of curves $-\Gamma_1 + \Gamma_2$:

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma_2} \frac{f(\xi)}{(\xi - z)}\, d\xi - \frac{1}{2\pi i} \int_{\Gamma_1} \frac{f(\xi)}{(\xi - z)}\, d\xi. \qquad (z \in D_2 \setminus \overline{D_1})$$

Then, by Lemma 1, we express $1/(\xi - z)$ firstly in terms of $\xi - c_n$ in Γ_2

$$\frac{1}{\xi - z} = \sum_{n \geq 0} \frac{(z - c_1)\ldots(z - c_n)}{(\xi - c_1)\ldots(\xi - c_{n+1})}, \qquad (\xi \in |\Gamma_2|)$$

and secondly in terms of $\xi - d_n$ in Γ_1

$$-\frac{1}{\xi - z} = \sum_{n \geq 0} \frac{(\xi - d_1)\ldots(\xi - d_n)}{(z - d_1)\ldots(z - d_{n+1})}. \qquad (\xi \in |\Gamma_1|)$$

Clearly, these series are locally uniformly absolutely convergent, so we can exchange integrating and summing to obtain the asserted expansion. $\qquad \square$

Observe that the representation of a holomorphic function as a series of form

$$\sum_{n \geq 1} \frac{a_{-n}}{(z - d_1)\ldots(z - d_n)} + \sum_{n \geq 0} a_n\, (z - c_1)\ldots(z - c_n)$$

is not unique when some d_n or c_n are outside the domain of z-s in which we consider such a representation.

For a holomorphic function $f : G \longrightarrow \mathbb{C}$ and any $n \geq 1$ we can define a holomorphic function $\Delta^{n-1}f : G^n \longrightarrow \mathbb{C}$ by

$$\Delta^{n-1}f(z_1, \ldots, z_n) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\xi)}{(\xi - z_1)\ldots(\xi - z_n)}\, d\xi,$$

where Γ is any cycle of curves for which $\mathrm{ind}_\Gamma(z_i) = 1$ and $\mathrm{ind}_\Gamma(z) = 0$ for $z \in \mathbb{C} \setminus G$.

Corollary 1. *Consider an open disc $D \subset \mathbb{C}$ with centre c and a holomorphic function $f : D \longrightarrow \mathbb{C}$. Consider also a sequence of complex numbers (c_n) such that $c_n \in D$ tends to c. Then the function f has the following representation as the sum of a locally uniformly absolutely convergent series:*

$$f(z) = \sum_{n \geq 0} \Delta^n f(c_1, \ldots, c_{n+1})\, (z - c_1)\ldots(z - c_n). \qquad (z \in D)$$

The coefficients are unique.

Proof. One only has to apply the Cauchy theorem to Theorem 1 for a sequence of open discs such that their closures have D as their union and each of which contains all of c_n-s. The uniqueness follows from property 7) below. $\qquad \square$

We revert now to the general domain G. We list several simple properties of the function Δ^n

(1) $f \mapsto \Delta^n f(c_1, \ldots, c_{n+1})$ is a linear functional.
(2) For any permutation σ of $\{1, \ldots, n\}$ we have

$$\Delta^{n-1} f(c_{\sigma(1)}, \ldots, c_{\sigma(n)}) = \Delta^{n-1} f(c_1, \ldots, c_n).$$

(3) $\Delta^n f(c, \ldots, c) = \frac{f^{(n)}(c)}{n!}$ (in particular $\Delta^0 f = f$).

(4) If $c_1, \ldots, c_n$ are all distinct then

$$\Delta^{n-1}f(c_1, \ldots, c_n) = \sum_{1 \le k \le n} \frac{f(c_k)}{(c_k - c_1) \ldots \widehat{(c_k - c_k)} \ldots (c_k - c_n)}$$

$$= \sum_{1 \le k \le n} \frac{f(c_k)}{(z \mapsto (z - c_1) \ldots (z - c_n))'(c_k)}.$$

(5) $\Delta^n f(c_1, \ldots, c_{n+1}) = \Delta^k \left[\Delta^{n-k} f(c_1, \ldots, c_{n-k}, \bullet) \right](c_{n-k+1}, \ldots, c_{n+1}).$

(6)

$\Delta^n f(c_1, \ldots, c_{n+1}) =$

$$= \begin{cases} \frac{\Delta^{n-1}f(c_1, \ldots, c_{n-1}, c_{n+1}) - \Delta^{n-1}f(c_1, \ldots, c_{n-1}, c_n)}{c_{n+1} - c_n}, & \text{when } c_n \ne c_{n+1}; \\ \left[\Delta^{n-1} f(c_1, \ldots, c_{n-1}, \bullet) \right]'(c_n), & \text{when } c_n = c_{n+1}. \end{cases}$$

(7) In the algebra of functions holomorphic on G, the value $\Delta^n f(c_1, \ldots, c_{n+1})$ is the leading coefficient of

$$f(z) \mod (z - c_1) \ldots (z - c_{n+1}),$$

that is to say if for a holomorphic function g we have

$$f(z) = g(z)(z - c_1) \ldots (z - c_{n+1}) + a_n z^n + \text{lower order terms},$$

say

$$f(z) = a_0 + a_1(z - c_1) + \cdots + a_n(z - c_1) \ldots (z - c_n) + g(z)(z - c_1) \ldots (z - c_{n+1})$$

then $\Delta^n f(c_1, \ldots, c_{n+1}) = a_n.$

Proof. 1,2) need no explanation, 3) is a well-known fact, 4) follows from the residue theorem.

The only not-so-trivial property is 5). In case when c_i are all distinct it follows from 4) by $n-k$ substitutions $s = n-k$, $a_q = c_i - c_{n-k+q}$, for each i from $\{1, \ldots, n-k\}$ in the following readily-obtainable equality for distinct nonzero $a_1, \ldots, a_s$

$$\frac{1/a_1}{(a_2 - a_1) \ldots (a_s - a_1)} + \ldots + \frac{1/a_s}{(a_1 - a_s) \ldots (a_{s-1} - a_s)} = \frac{1}{a_1 \ldots a_s}.$$

(This equality can be proved in a few ways. A. By the residue theorem and the following equality

$$\int_\Gamma \frac{d\xi}{\xi(\xi - a_1) \ldots (\xi - a_s)} = 0, \quad \text{for large circles } \Gamma.$$

B. By combining 4) for $f(z) = 1/z$, Lemma 1 and the uniqueness of decomposition in Collorary 1, or C. Simply by induction.) The general case follows by the identity theorem in G^{n+1}.

Now, property 6) can be obtained by combining 5) with 4), or 3) depending on case. Finally, one more use of the residue theorem gives 7). $\qquad \square$

Motivated by 5) above, we call $\Delta^n f(c_1, \ldots, c_{n+1})$ the *n-th devided difference* of f at points $c_1, \ldots, c_{n+1}$.

3. Is this a known result?

In the few sources on interpolation I had the occasion and opportunity to browse through I haven't actually recognized this result. However, because of quite a few similar (yet not quite there) theorems I thought it must be known and haven't tried to publish it.

Now, after a few years of not having anything to do with complex analysis, hoping that at least the method of proof has some new elements to it, I am placing it on arXiv with the expectation that someone more knowledgable on the subject will provide me with some valuable feedback.

There is a vast room for improvement in the statement of the theorem and in all the theory that might surround it. But there would be no point in going into it if it all had been considered before.

References

[1] Davis, P. (1975) *Interpolation and Approximation*, Dover
[2] Nörlund, N. (1926) *Leçsons sur le Séries d'Interpolation*, Gauthier-Villars

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